OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

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CHAPTER-6 : COMPLEX NUMBERS

UNIT TEST-1

1. Prove that the general value of x which satisfies the equation $(\cos x + i\sin x)(\cos 3x + i\sin 3x)$ $(\cos 5x + i\sin 5x) \ldots [\cos(2n-1)$ $x + i\sin(2n-1)x] = 1$ is $\dfrac{2r\pi}{n^2}$:

2. Let $f_p(\beta) = \left(\cos\dfrac{\beta}{p^2} + i\sin\dfrac{\beta}{p^2}\right)\left(\cos\dfrac{2\beta}{p^2} + i\sin\dfrac{2\beta}{p^2}\right)\ldots$ $\left(\cos\dfrac{\beta}{p} + i\sin\dfrac{\beta}{p}\right)$ find $\lim\limits_{n\to\infty} f_n(\pi) =$?

3. Prove that $\left(\dfrac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi}\right) = \cos n\phi + i\sin n\phi$

4. Prove that If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, then $\alpha^n + \beta^n = 2^{n+1}$ $\cos(n\pi/3)$:

5. Prove that :
If $1, \omega, \omega^2$ are the three cube roots of unity, then
$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$
$= (2a - b - c)(2b - c - a)(2c - a - b)$
$= 27\,abc$ if $a + b + c = 0$.

6. Prove that :
If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$,
then $\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0$
$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0$

7. Let z be a complex number satisfying $|z - 5i| \le 1$ such that amp z is minimum. Then z is equal to $\dfrac{2\sqrt{6} + 24i}{5}$ prove.

8. If a, b, c are distinct integers and $\omega \ne 1$ is a cube root of unity. Find minimum value of $x = |a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$:

9. If z is a complex number having least absolute value and $|z - 2 + 2i| = 1$, then find 2.

10. If $|z - 25i| \le 15$, then prove that :
$|$ max, amp (z) – min. amp (z) $| = \pi - 2\cos^{-1}\left(\dfrac{3}{5}\right)$

11. Find the least value of r for which the two curves arg $(z) = \dfrac{\pi}{6}$ and $|z - 2\sqrt{3}i| = r$ intersect is :

12. If $\alpha + i\beta = \tan^{-1}(z)$, $z = x + iy$ and α is constant, then prove that locus of z is $x^2 + y^2 + 2x\cot 2\alpha = 1$.

13. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is a complex number such that the argument of $\dfrac{(z - z_1)}{(z - z_2)}$ is $\dfrac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

14. For any two non-zero complex numbers z_1, z_2 then prove the inequality
$$(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right| \le 2(|z_1| + |z_2|).$$

15. If x, y, z are three distinct complex numbers and a, b, c are three + ive real numbers satisfying the relation $\dfrac{a}{|y - z|} = \dfrac{b}{|z - x|} = \dfrac{c}{|x - y|}$ then prove that $\dfrac{a^2}{y - z} + \dfrac{b^2}{z - x} + \dfrac{c^2}{x - y} = 0$.

Hints and Solutions

1. (a) $E = e^{ix}e^{3ix}e^{5ix} \ldots e^{(2n-1)x} = 1$

$= e^{in^2x} = 1$

By sum of A.P., $x + 3x + 5x + \ldots n$ terms

or $\cos n^2x + i\sin n^2x = 1$

or Equating real and imaginary parts,

$\cos n^2x = 1 = \cos 0°$

$\therefore \quad n^2 x = 2r\pi \pm 0$

$\therefore \quad x = \dfrac{2r\pi}{n^2} \Rightarrow$ (d)

2. (c) In the last term write $\dfrac{\beta}{p}$ as $\dfrac{p\beta}{p^2}$

$\therefore \; f_p(\beta) = e^{i\beta/p^2 + 2i\beta/p^2 + \ldots + p\, i\beta/p^2}$

$$= e^{i\beta/p^2 \{1+2+3+\ldots+p\}} = e^{i\beta/p^2 \left\{\frac{p(p+1)}{2}\right\}}$$

$$= e^{\frac{i\beta}{2}\left(1+\frac{1}{p}\right)}$$

Now put $p = n, \beta = \pi$

$$f_n(\pi) = e^{\frac{i\pi}{2}\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} f_n(\pi) = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

3. L.H.S. $= \left[\dfrac{2\cos^2(\phi/2) + 2i\sin(\phi/2)\cos(\phi/2)}{2\cos^2\phi - 2i\sin(\phi/2)\cos(\phi/2)}\right]^n$

$$= \left[\frac{\cos(\phi/2) + i\sin(\phi/2)}{\cos(\phi/2) - i\sin(\phi/2)}\right]^n$$

$$= \left\{\frac{e^{i(\phi/2)}}{e^{-i(\phi/2)}}\right]^n = (e^{i\phi})^n$$

$$e^{in\phi} = \cos n\phi + i\sin n\phi$$

$$x^2 - 2x + 4 = 0$$

4. (a) $\therefore \; x + 1 \pm i\sqrt{3} = r(\cos\theta \pm i\sin\theta)$

where $r = 2$, $\theta = \pi/3$

$\therefore \quad \alpha^n + \beta^n = r^n[\cos n\theta + i\sin n\theta + \cos n\theta - \sin n\theta]$

$$r^n \cdot 2\cos n\theta = 2^{n+1}\cos(n\pi/3)$$

5. (a) $x^3 + y^3 = (x+y)(\omega x + \omega^2 y)(\omega^2 x + \omega y)$

Each of the factor corresponds to three given factors. In case $a+b+c = 0$ *i.e.*, $b + c = -a$ etc. Then the answer is $(3a)\,(3b)\,(3c) = 27abc$.

6. (a) $a = \cos\alpha + i\sin\alpha$

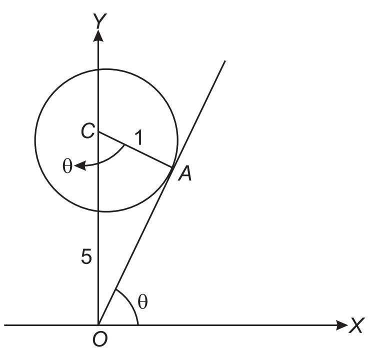
$$a + b + c = 0$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a^{-1} + b^{-1} + c^{-1}$$

$$= \Sigma\cos\alpha - i\,\Sigma\sin\alpha = 0$$

$\therefore \quad bc + ca + ab = 0$ or $e^{i(\beta+\gamma)} + e^{i(\gamma+\alpha)} + e^{i(\alpha+\beta)} = 0$.

7. (a)



$|z - 5i| \le 1$ represent all points lying inside and on the circle centred at (0, 5) and of radius 1. Clearly the point A has minimum amplitude $\theta = \angle AOX = \angle ACO$, $OA^2 = 25 - 1 = 24$

$\therefore \quad \cos\theta = \dfrac{1}{5}$

$\Rightarrow \quad \sin\theta = \dfrac{2\sqrt{6}}{5}$

$$x = OA\cos\theta = 2\sqrt{6}\cdot\frac{1}{5}$$

$$y = OA\sin\theta$$

$$= 2\sqrt{6}\cdot\frac{2\sqrt{6}}{5}$$

$\therefore \; A$ is $z = x + iy = \dfrac{2\sqrt{6}}{5}(1 + 2\sqrt{6}\,i)$

8. ($6\sqrt{2}$) Let $z = a + bw + cw^2$

$\therefore \quad \bar{z} = a + bw^2 + cw$

$\therefore \quad |z| = |\bar{z}| \quad$ ...(1)

and $\quad z\bar{z} = a^2 + b^2 + c^2 - bc - ca - ab$

$$|z|^2 = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

$x = |z| + |\bar{z}| = 2|z| \quad$ by (1)

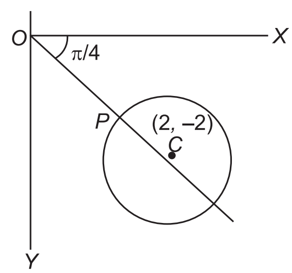
$\therefore \quad x^2 = 4|z|^2 = 4\cdot\dfrac{1}{2}[\Sigma(a-b)^2]$

$\therefore \quad x = \sqrt{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \quad$ ...(2)

Since, a, b, c are integers, x will be minimum if a, b, c are consecutive integers $p, p+1, p+2$.

$\therefore \quad x = \sqrt{2}[1 + 1 + 4] = 6\sqrt{2}$

9. (a)



The given equation represents points on circle whose centre is C (2, –2) in 4th quadrant and radius 1 where OC is inclined at angle of 45° to x-axis. Clearly the point P on OC will have least absolute value, say r.

$$r = OP = OC - PC = \sqrt{4+4} - 1 = 2\sqrt{2} - 1$$

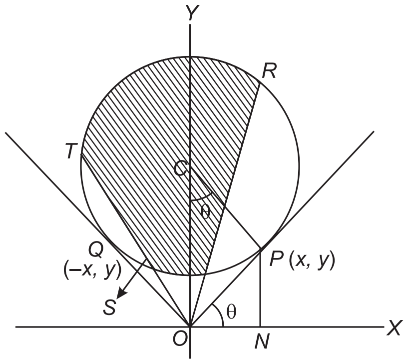
$\therefore$ Point P is $(r\cos 45°, -r\sin 45°)$

$$= \left(\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}\right)$$

$$= \frac{r}{\sqrt{2}}(1 - i)$$

$\therefore \quad z = \left(2 - \dfrac{1}{\sqrt{2}}\right)(1 - i)$

10. (a)



Point P has the min. amplitude θ and the corresponding point Q has maximum amplitude $\pi - \theta$ where

$$\cos\theta = \frac{CP}{OC} = \frac{15}{25} = \frac{3}{5},\ \sin\theta = \frac{4}{5}$$

$$\pi - \theta - \theta = \pi - 2\theta$$

$$\therefore \quad \pi - 2\cos^{-1}\left(\frac{3}{5}\right).$$

11. (a) Change the two equations to Cartesian form :

$$\arg z = \frac{\pi}{6} \Rightarrow \frac{y}{x} = \tan 30° \Rightarrow x - \sqrt{3}y = 0$$

$|z - 2\sqrt{3}i| = r$ represents a circle centred at $(0, 2\sqrt{3})$ and radius r.

$$x^2 + (y - 2\sqrt{3})^2 = r^2$$

If the two curves touch, then $p = r$ and in case they intersect, then $p \le r$ when p is perpendicular form centre $(0,\ 2\sqrt{3})$ to line

$$\therefore \quad \left|\frac{0 - 2\sqrt{3}\cdot\sqrt{3}}{2}\right| \le r \text{ or } 3 \le r$$

$$\therefore \quad r \ge 3$$

Hence the least value of r is 3 for the curves to intersect. (Touching is also intersection at two coincident points).

12. (a) The first equation represents points on a circle with centre at $(-\sqrt{2}, 0)$ and radius $r_1 = \sqrt{a^2 - 3a - 2}$ (Real). the second equation represents points within a circle centred at $(0,\ \sqrt{2})$ and radius a.

Since both hold for at least one point therefore the two circles must intersect and as such

$$C_1C_2 < r_1 + r_2$$

or $\quad \sqrt{2+2} < \sqrt{a^2 - 3a - 2} + a$

or $\quad (2 - a)^2 < a^2 - 3a - 2$

or $\quad -4a + 4 < -3a - 2$ or $6 < a$

$\therefore \quad a > 6 \Rightarrow$ (a)

13. (a) $\tan(\alpha + i\beta) = x + iy$

$\therefore\ \tan(\alpha - i\beta) = x - iy$ conjugate

α is constant and β is unknown to be eliminated

$$\tan 2\alpha = \tan(\overline{\alpha + i\beta} + \overline{\alpha - i\beta})$$

$$\tan 2\alpha = \frac{x + iy + x - iy}{1 - (x^2 + y^2)}$$

$$1 - (x^2 + y^2) = 2x\cot 2\alpha$$

$$\therefore \quad x^2 + y^2 + 2x\cot 2\alpha = 1$$

14. (a) $\frac{\pi}{4} = \arg\frac{z - z_1}{z - z_2} = \arg\left(\frac{x + iy - 10 - 6i}{x + iy - 4 - 6i}\right)$

$$\frac{\pi}{4} = \arg\frac{z - z_1}{z - z_2} = \arg(z - z_1) - \arg(z - z_2)$$

or $\frac{\pi}{4} = \theta_{1;} - \theta_2$

or $\tan^{-1} 1 = \tan^{-1}\frac{y_1}{x_1} - \tan^{-1}\frac{y_2}{x_2}$

or $\tan^{-1} 1 = \tan^{-1}\frac{y - 6}{x - 10} - \tan^{-1}\frac{y - 6}{x - 4}$

or $\tan^{-1} 1 = \tan^{-1}\dfrac{(y - 6)\left[\frac{1}{x - 10} - \frac{1}{x - 4}\right]}{1 + \frac{(y - 6)^2}{(x - 10)(x - 4)}}$

$$\therefore \quad 1 = \frac{6(y - 6)}{x^2 - 14x + 40 + y^2 - 12y + 36}$$

or $\quad x^2 - 14x + y^2 - 18y + 112 = 0$

or $(x - 7)^2 - 49 + (y - 9)^2 - 81 + 112 = 0$

$\therefore \quad (x - 7)^2 + (y - 9)^2 = 18$ . ..(1)

$$\therefore \quad |z - 7 - 9i| = |(x - 7) + i(y - 9)|$$

$$= [(x - 7)^2 + (y - 9)^2]^{1/2}$$

$$= \sqrt{(18)} = 3\sqrt{2}$$

15. (a) We have :

$$(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right| \le (|z_1| + |z_2|)$$

$$\left[\left|\frac{z_1}{|z_1|}\right| + \left|\frac{z_2}{|z_2|}\right|\right]$$

$$= (|z_1| + |z_2|)\left[\frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|}\right] = 2(|z_1| + |z_2|)$$

16. (a) Let $\frac{a}{|y - z|} = \lambda$, then

$$a^2 = \lambda^2|y - z|^2 = \lambda^2(y - z)\overline{(y - z)}$$

$$\therefore \quad \frac{a^2}{y - z} = \lambda^2(\bar{y} - \bar{z})$$

$$\therefore \quad \Sigma\frac{a^2}{y - z} = \lambda^2\,\Sigma(\bar{y} - \bar{z}) = 0$$